SwInBee

Practice exercises — worked solutions

Solutions to the exercises in the SwInBee preparation document.

3 The substitution method

3.1 $\displaystyle\int \sin x\, \cos(\cos x) \sin(\sin(\cos x))\, dx$ SwInBee 2022

Substitute one layer at a time, starting from the innermost function. With $w = \cos x$ we have $dw = -\sin x\, dx$:

$$\int \sin x \cos(\cos x) \sin(\sin(\cos x))\, dx = -\int \cos w\, \sin(\sin w)\, dw.$$

Now with $s = \sin w$ we have $ds = \cos w\, dw$:

$$-\int \sin s\, ds = \cos s + C = \cos(\sin w) + C = \cos(\sin(\cos x)) + C.$$

Answer: $\cos(\sin(\cos x)) + C$

3.2 $\displaystyle\int_0^9 \frac{dx}{\sqrt{1+\sqrt{x}}}$ SwInBee 2019

Substitute for the outer root, $t^2 = 1 + \sqrt{x}$. Then $\sqrt{x} = t^2 - 1$, $x = \left(t^2-1\right)^2$ and $dx = 4t\left(t^2-1\right) dt$. For the limits, $x = 0$ gives $t = 1$, and $x = 9$ gives $\sqrt{x} = 3$, so $t = 2$:

$$\int_0^9 \frac{dx}{\sqrt{1+\sqrt{x}}} = \int_1^2 \frac{4t\left(t^2-1\right)}{t}\, dt = 4\int_1^2 \left(t^2-1\right) dt = 4\left[\frac{t^3}{3} - t\right]_1^2 = 4\left(\frac{2}{3} + \frac{2}{3}\right) = \frac{16}{3}.$$

Answer: $\dfrac{16}{3}$

Because we changed the limits, there was no need to substitute back.

3.3 $\displaystyle\int \frac{dx}{\sqrt{x^2-2024}}$ SwInBee 2024

This is a quadratic under a square root, of the form $\sqrt{x^2-a^2}$ with $a = \sqrt{2024}$. Put $x = \sqrt{2024}\cosh t$, so that $dx = \sqrt{2024}\sinh t\, dt$ and, since $\cosh^2 t - 1 = \sinh^2 t$, $\sqrt{x^2-2024} = \sqrt{2024}\sinh t$:

$$\int \frac{dx}{\sqrt{x^2-2024}} = \int \frac{\sqrt{2024}\sinh t}{\sqrt{2024}\sinh t}\, dt = t + C = \cosh^{-1}\left(\frac{x}{\sqrt{2024}}\right) + C.$$

Answer: $\cosh^{-1}\left(\dfrac{x}{\sqrt{2024}}\right) + C$

or $\ln\left(x + \sqrt{x^2-2024}\right) + C$

The two forms differ by the constant $\ln\sqrt{2024}$, so both are correct. This integral is also in the table of standard integrals.

3.4 $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$ SwInBee 2019

This has the form $\sqrt{a^2-x^2}$ with $a=2$, so put $x=2\sin\theta$. Then $dx=2\cos\theta\,d\theta$ and $\sqrt{4-x^2}=2\cos\theta$, and the limits become $\theta=0$ and $\theta=\pi/2$:

$$\int_0^2 \sqrt{4-x^2}\,dx = \int_0^{\pi/2} 4\cos^2\theta\,d\theta = 2\int_0^{\pi/2} (1+\cos 2\theta)\,d\theta = 2\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \pi.$$

Answer: π

In the second step we used $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$. As a check, the integral is the area of a quarter of a circle of radius 2, which is $\frac{1}{4}\pi\cdot 2^2=\pi$.

4 Partial fraction decomposition

4.1 $\displaystyle\int \frac{dx}{x^2+5x+6}$ SwInBee 2022

The denominator factors as $(x+2)(x+3)$, so we write $\frac{1}{(x+2)(x+3)} = \frac{A}{x+2}+\frac{B}{x+3}$. Multiplying through by $(x+2)(x+3)$ gives $1=A(x+3)+B(x+2)$. Setting $x=-2$ gives $A=1$, and setting $x=-3$ gives $B=-1$, so

$$\int \frac{dx}{x^2+5x+6} = \int\left(\frac{1}{x+2}-\frac{1}{x+3}\right)dx = \ln|x+2|-\ln|x+3|+C.$$

Answer: $\ln|x+2|-\ln|x+3|+C$

This can also be written as $\ln\left|\frac{x+2}{x+3}\right|+C$.

4.2 $\displaystyle\int \frac{2x^3}{5+3x^2}\,dx$ SwInBee 2024

The numerator has a higher degree than the denominator, so we divide first:

$$\frac{2x^3}{3x^2+5} = \frac{2x}{3} - \frac{10x}{3\,(3x^2+5)}.$$

The numerator of the second term is a multiple of $6x$, the derivative of the denominator, so it integrates to a logarithm:

$$\int \frac{2x^3}{5+3x^2}\,dx = \frac{x^2}{3} - \frac{10}{3}\cdot\frac{1}{6}\ln\left(3x^2+5\right)+C = \frac{x^2}{3}-\frac{5}{9}\ln\left(3x^2+5\right)+C.$$

Answer: $\dfrac{x^2}{3}-\dfrac{5}{9}\ln\left(3x^2+5\right)+C$

No absolute value is needed, since $3x^2+5>0$.

4.3 $\displaystyle\int \frac{4x+1}{1+x^2}\,dx$ SwInBee 2024

The denominator is an irreducible quadratic, so we split the numerator into a multiple of $2x$, the derivative of the denominator, plus a constant:

$$\frac{4x+1}{1+x^2} = 2\cdot\frac{2x}{1+x^2}+\frac{1}{1+x^2}.$$

Then

$$\int \frac{4x+1}{1+x^2}\,dx = 2\ln\left(1+x^2\right) + \arctan x + C.$$

Answer: $2\ln\left(1+x^2\right) + \arctan x + C$

A quadratic denominator typically gives a logarithm plus an $\arctan$. If the denominator is not already of the form x^2+a^2, complete the square first.

4.4 $\displaystyle\int \frac{dx}{x^3\left(x^2+1\right)}$ SwInBee 2025

The full decomposition, $\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{Dx+E}{x^2+1}$, has five constants to find. It is quicker to add and subtract x^2 in the numerator:

$$\frac{1}{x^3\left(x^2+1\right)} = \frac{\left(1+x^2\right)-x^2}{x^3\left(x^2+1\right)} = \frac{1}{x^3} - \frac{1}{x\left(x^2+1\right)}.$$

The second term decomposes as $\frac{1}{x}-\frac{x}{x^2+1}$. So

$$\int \frac{dx}{x^3\left(x^2+1\right)} = \int \left(\frac{1}{x^3}-\frac{1}{x}+\frac{x}{x^2+1}\right)dx = -\frac{1}{2x^2} - \ln|x| + \frac{1}{2}\ln\left(x^2+1\right) + C.$$

Answer: $-\dfrac{1}{2x^2} - \ln|x| + \dfrac{1}{2}\ln\left(x^2+1\right) + C$

Writing $1 = (1+x^2) - x^2$ makes life a lot easier.

5 Integration by parts

5.1 $\displaystyle\int x^2 \ln x\,dx$ SwInBee 2022

Take $u = \ln x$, which becomes simpler when differentiated, and $dv = x^2\,dx$. Then $du = dx/x$ and $v = x^3/3$:

$$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C.$$

Answer: $\dfrac{x^3}{3}\ln x - \dfrac{x^3}{9} + C$

The other choice, $dv = \ln x\,dx$, needs the integral of $\ln x$ and leads to a harder integral than the original. If integration by parts makes things more complicated, try swapping u and dv.

5.2 $\displaystyle\int \frac{4x}{e^{2x+3}}\,dx$ SwInBee 2024

Write the integral as $4\int xe^{-2x-3}\,dx$, and take $u = x$ and $dv = e^{-2x-3}\,dx$, so that $v = -\frac{1}{2}e^{-2x-3}$:

$$\begin{aligned} 4\int xe^{-2x-3}\,dx &= 4\left(-\frac{x}{2}e^{-2x-3} + \frac{1}{2}\int e^{-2x-3}\,dx\right) \\ &= -2xe^{-2x-3} - e^{-2x-3} + C = -\left(2x+1\right)e^{-2x-3} + C. \end{aligned}$$

Answer: $-\left(2x+1\right)e^{-2x-3} + C$

Take care with the factor of $-\frac{1}{2}$ in v. Differentiating the answer is a quick way to catch this kind of mistake.

5.3 $\int x^2 \arcsin x \, dx$ SwInBee 2023

Take $u = \arcsin x$, which becomes simpler when differentiated, and $dv = x^2\,dx$. Then $du = dx/\sqrt{1-x^2}$ and $v = x^3/3$:

$$\int x^2 \arcsin x \, dx = \frac{x^3}{3}\arcsin x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx.$$

The remaining integral needs a substitution. With $w = 1 - x^2$ we have $dw = -2x\,dx$ and $x^2 = 1 - w$, so

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1-w}{\sqrt{w}}\,dw = -\frac{1}{2}\left(2\sqrt{w} - \frac{2}{3}w^{3/2}\right) = -\sqrt{w} + \frac{w^{3/2}}{3}.$$

Substituting back $w = 1 - x^2$ and simplifying,

$$\begin{aligned}\int x^2 \arcsin x \, dx &= \frac{x^3}{3}\arcsin x + \frac{\sqrt{1-x^2}}{3} - \frac{\left(1-x^2\right)^{3/2}}{9} + C\\ &= \frac{x^3}{3}\arcsin x + \frac{\left(x^2+2\right)\sqrt{1-x^2}}{9} + C.\end{aligned}$$

Answer: $\dfrac{x^3}{3}\arcsin x + \dfrac{\left(x^2+2\right)\sqrt{1-x^2}}{9} + C$

Integration by parts to remove an inverse function, followed by a substitution, is a common combination.

5.4 $\int (\ln x)^3 \, dx$ SwInBee 2022

Take $dv = dx$. Each application of parts reduces the power of the logarithm by one:

$$\begin{aligned}\int (\ln x)^3\,dx &= x(\ln x)^3 - 3\int (\ln x)^2\,dx,\\ \int (\ln x)^2\,dx &= x(\ln x)^2 - 2\int \ln x\,dx,\\ \int \ln x\,dx &= x\ln x - x.\end{aligned}$$

Combining these,

$$\int (\ln x)^3\,dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C.$$

Answer: $x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$

In general, $\int (\ln x)^n\,dx = x\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}(\ln x)^k$. The 2019 paper asked for $\int_1^3 (\ln x)^4\,dx$.

6 Tricky integrals

6.1 $\int \exp(5\ln x)\,dx$ SwInBee 2023

The integrand simplifies before we integrate:

$$\exp(5\ln x) = \exp\left(\ln x^5\right) = x^5, \qquad \text{so} \qquad \int \exp(5\ln x)\,dx = \frac{x^6}{6} + C.$$

Answer: $\dfrac{x^6}{6} + C$

Always check whether the integrand simplifies first!

6.2 $\displaystyle\int_{-\pi}^{\pi} x^{11} \sin^3\left(x^2\right) dx$ SwInBee 2024

The limits are symmetric about 0, so we check whether the integrand is odd. The factor x^{11} is odd, and $\sin^3\left(x^2\right)$ is even since it is a function of x^2. So the integrand is odd, and

$$\int_{-\pi}^{\pi} x^{11} \sin^3\left(x^2\right) dx = 0.$$

Answer: 0

An antiderivative can be found, by substituting $u = x^2$ and then using integration by parts several times, but symmetry gives the answer immediately.

6.3 $\displaystyle\int_0^{1/6} \left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \cdots\right) dx$ SwInBee 2019

First we sum the series. It is the derivative of the geometric series $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$, which converges for $|x| < 1$, so

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}.$$

The interval $[0, \frac{1}{6}]$ lies inside $|x| < 1$, so

$$\int_0^{1/6} \frac{dx}{(1-x)^2} = \left[\frac{1}{1-x}\right]_0^{1/6} = \frac{1}{5/6} - 1 = \frac{6}{5} - 1 = \frac{1}{5}.$$

Answer: $\dfrac{1}{5}$

Alternatively, integrate term by term: $\int_0^{1/6} \sum n x^{n-1}\, dx = \sum \left(\frac{1}{6}\right)^n = \frac{1/6}{1-1/6} = \frac{1}{5}$.

6.4 $\displaystyle\int_0^1 \frac{x^4 (1-x)^4}{1+x^2}\, dx$ SwInBee 2024 and 2025

Expand the numerator and use long division to divide by $1 + x^2$:

$$\frac{x^4(1-x)^4}{1+x^2} = \frac{x^8 - 4x^7 + 6x^6 - 4x^5 + x^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}.$$

Each term can now be integrated:

$$\begin{aligned}\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx &= \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\arctan x\right]_0^1 \\ &= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi = \frac{22}{7} - \pi.\end{aligned}$$

Answer: $\dfrac{22}{7} - \pi$

Since the integrand is positive on $(0, 1)$, this shows that $\pi < \frac{22}{7}$. The integral is long, but it does not need any special insight.